Exact Thresholds and Instanton Effects in String Theory [*]

N.A. Obers[a†] and B. Pioline[b‡]

[a] *Spinoza Institute, Utrecht University, 3584 CE Utrecht, The Netherlands*

and

Institute for Theoretical Physics, Utrecht University, 3508 TA Utrecht, The Netherlands

[b] *Jefferson Physical Laboratory, Harvard University, Cambridge MA 02138, USA*

and

LPTHE, Université Pierre et Marie Curie, Paris VI

Université Denis Diderot, Paris VII, Boîte 126, Tour 16, 1ᵉʳ étage, 4 place Jussieu,

F-75252 Paris Cedex 05, France

In this lecture we summarize some recent work on the understanding of instanton effects in string theories with 16 supersymmetries. In particular, we consider F^4 couplings using the duality between the heterotic string on T^4 and type IIA on K_3 at an orbifold point, as well as higher and lower dimensional versions of this string-string duality. At the perturbative level a non-trivial test of the duality, requiring several miraculous identities, is presented by matching a purely one-loop heterotic amplitude to a purely tree-level type II result. A wide variety of non-perturbative effects is shown to occur in this setting, including D-brane instantons for type IIA on $K_3 \times S^1$ and NS5-brane instantons for type IIA on $K_3 \times T^2$. Moreover, the analysis of the three-dimensional case, which possesses a non-perturbative $SO(8, 24, \mathbb{Z})$ U-duality, reveals the presence of Kaluza-Klein 5-brane instanton effects, both on the heterotic and the type II side.

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[†] `obers@phys.uu.nl`

[‡] `pioline@physics.harvard.edu`

1 Introduction

The discovery of non-perturbative dualities and symmetries of string theory over the last five years, has opened up the possibility of obtaining quantities, such as spectrum and amplitudes, that are exact for all values of the string coupling g_s. In particular, the computation of exact amplitudes in string theory has on the one hand allowed to test the various conjectured dualities, and on the other hand enabled us to compute for the first time instanton corrections to the effective action. This in turn makes it possible to address the challenging problem of understanding the rules of semi-classical instanton calculus in string theory.

One of the central ingredients in this development has been supersymmetry, which protects special "BPS" states in the spectrum, invariant under part of the supersymmetry charges, from quantum corrections. Likewise, some special "BPS-saturated" terms in the effective action do not receive perturbative corrections beyond a certain order (typically one-loop). Imposing in addition invariance under the duality symmetry groups, often allows to determine these amplitudes in terms of automorphic forms, which generalize the familiar $Sl(2, Z)$ modular forms to higher arithmetic groups.

The coupling $f_{\mathcal{O}}$ to a higher dimension operator $\mathcal{O}$ in the effective action of some compactified string theory is in general some function on the moduli space, which includes the dilaton, along with the moduli describing the size and shape of the compactification manifold, as well as possible other background fields (e.g. the vevs of the RR gauge potentials). For a BPS-saturated amplitude this coupling has a weak-coupling expansion of the form

$$f_{\mathcal{O}} \sim \text{perturbative} + \text{instanton} \tag{1.1}$$

with only a finite number of perturbative terms. From the non-perturbative part, describing the contributions of instantons, one may then read off the semiclassical configurations that are present, and e.g. the corresponding instanton measure.

For the methodology behind the computation of exact amplitudes it is useful to distinguish between two types of dualities. The first, and most powerful, type is duality *symmetry*, where distinct regimes of the the same theory are related. This is referred to as U-duality and includes perturbative T-duality as well as symmetries relating strong with weak string coupling. Examples for theories with 32 supersymmetries include the $Sl(2, \mathbb{Z})$ duality of (uncompactified) type IIB string theory, and the more general $E_{d+1(d+1)}(\mathbb{Z})$ duality symmetry of type II theory on the d-torus T^d (or equivalently, M-theory on T^{d+1}) [1, 2, 3] (see [4] for a review of U-duality). In the case of toroidal compactification of the heterotic string, which has 16 supersymmetries, it is also known that the $SO(d, d+16, \mathbb{Z})$

T-duality symmetry for $d \leq 5$ is enhanced to a non-perturbative $SO(6, 22, \mathbb{Z}) \times Sl(2, \mathbb{Z})$ for $d = 6$ and $SO(8, 24, \mathbb{Z})$ for $d = 7$. Finally, type IIB on K_3 has an $SO(5, 21, \mathbb{Z})$ U-duality symmetry. These cases are to be contrasted with string-string dualities, which relate two different string theories in distinct regimes. The most prominent examples, all having 16 supersymmetries, are heterotic-type I duality [5], which already holds in ten dimensions, as well as the duality between heterotic on T^4 and type IIA on K_3 [6].

When the theory has a U-duality symmetry, supersymmetry constraints in the form of second order differential equations together with constraints coming from duality invariance and the known leading perturbative behavior have been used to obtain exact BPS-saturated couplings. The paradigm in theories with 32 supersymmetries has been the R^4 coupling, which e.g. for ten-dimensional type IIB has been shown to be equal to a certain (non-holomorphic) form of weight 0 of the S-duality group, namely the Eisenstein series of order $3/2$ [7, 8]. A perturbative expansion of the function exhibits the known tree and one-loop contribution, as well as an infinite series of exponentially suppressed terms attributed to D-instantons. This was generalized for toroidal compactifications of type II theory to Eisenstein series of the corresponding U-duality groups [9, 10, 11, 12].

In the case of string-string duality one cannot resort to such methods, but one has to depend on non-renormalization theorems stating that certain couplings are perturbatively (typically one-loop) exact. Under the duality these couplings then translate into non-perturbative results on the dual side, mapping world-sheet instantons to space-time instantons. The canonical examples, in theories with 16 supersymmetries, are the R^2 coupling [13, 14], which is believed to be one-loop exact in type II on $K_3 \times T^2$, and the F^4 coupling [15, 16, 17, 18, 19, 20], believed to be one-loop exact in toroidally compactified heterotic string up to four dimensions.

An interesting hybrid case consists of the duality in three dimensions between heterotic on T^7 and type IIA on $K_3 \times T^3$, each of which also has an $SO(8, 24, \mathbb{Z})$ U-duality symmetry [21]. The latter can be used to covariantize the one-loop heterotic result into an exact expression [22], which can then be used in turn to obtain to exact threshold on the type IIA side using string-string duality.

In this lecture, mainly based on [20, 22], we focus on the last two cases above, referring to [23] for a summary of the use of U-duality symmetry and Eisenstein series in determining the exact R^4 couplings in toroidal compactifications of type II strings and M-theory. Section 2 discusses the moduli space of heterotic string theory on T^d and its relation via triality to type II string theory on $K_3 \times T^{d-4}$. In Section 3 we present a non-trivial test of heterotic-type IIA duality by matching the one-loop F^4 heterotic amplitude in six dimensions to the dual tree-level type IIA result. Then in Section 4, we obtain from the F^4 amplitude in

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$D = 4$ and 5, non-perturbative D- and NS5-brane instanton effects in type IIA. Finally, Section 5 discusses the special three-dimensional case in which one finds instanton effects on both of the dual sides, revealing in particular heterotic 5-brane instantons as well as KK5-brane instantons on either side.

2   Heterotic and IIA moduli spaces and triality

The comparison of spectrum and amplitudes in two dual string theories requires a proper understanding of the corresponding moduli spaces and the precise mapping between them. Therefore, we start by recalling some basic facts about the heterotic moduli space for toroidal compactifications. For simplicity we consider mainly compactifications on square tori, even though the full duality becomes apparent only when including the Wilson lines and vielbein moduli on the heterotic side and the Ramond backgrounds on the type-II side.

We first consider the $E_8 \times E_8$ or $SO(32)$ heterotic string theory compactified on a torus T^d. For $d \leq 5$, the moduli space takes the form

$$\mathbb{R}^+ \times [SO(d) \times SO(d+16)]\backslash SO(d, d+16, \mathbb{R})/SO(d, d+16, \mathbb{Z}) , \tag{2.1}$$

where the first factor is parameterized by the T-duality invariant dilaton ϕ_{10-d} related to the ten-dimensional heterotic coupling g_{H} by $e^{-2\phi_{10-d}} = V_d/(g_{\mathrm{H}}^2 l_{\mathrm{H}}^d)$, with V_d the volume of the d-torus; the second factor is the standard Narain moduli space, describing the metric g and B-field b on the torus T^d, together with the Wilson lines y of the 16 U(1) gauge fields (in the Cartan torus of the ten-dimensional gauge group) along the d circles of the torus. The right action of the discrete group $SO(d, d+16, \mathbb{Z})$ reflects the invariance under T-duality. This moduli space can be parameterized by a viel-bein $e_{\mathrm{H}} \in SO(d, d+16, \mathbb{R})$ in the upper-triangular (Iwasawa) gauge. In particular, the dilatonic moduli parameterizing the Abelian (diagonal) part of the vielbein are the radii of compactification

$$x_{i=1\ldots d} = (R_1/l_{\mathrm{H}}, \ldots, R_d/l_{\mathrm{H}}) . \tag{2.2}$$

In order to determine the mapping of moduli to the dual descriptions, we will compare the BPS mass formula on both sides. For perturbative heterotic BPS states, it is simply given by

$$\mathcal{M}^2 = \frac{1}{l_{\mathrm{H}}^2} Q^t (M_{d,d+16} - \eta) Q , \tag{2.3}$$

where $Q = (m^i, q^I, n_i)$ is the vector of momenta, charges and windings and $M_{d,d+16} = e_{\mathrm{H}}^t e_{\mathrm{H}}$ in terms of the viel-bein. The charges $q^I, I = 1 \ldots 16$ take values in the even self dual lattice $E_8 \oplus E_8$ or D_{16}.

This description of the moduli space is quite complete for compactification down to 5 dimensions. For $D = 4$, however, the $\mathbb{R}^+$ factor in (2.1) is enhanced to $U(1)\backslash Sl(2,\mathbb{R})$, parameterized by a complex parameter $S = a + i/g_{4H}^2$, where $1/g_{4H}^2 = V_6/(g_H^2 l_H^6)$ is the four-dimensional string coupling constant and a the scalar dual to the Neveu-Schwarz two-form in four dimensions. The action of $Sl(2,\mathbb{Z})$ on S is conjectured to be a non-perturbative symmetry [24] of the heterotic string compactified on T^6. Further enhancement in $D = 3$ is due to the dualization of the 30 $U(1)$ gauge fields into scalars, which are then unified together with the dilaton and $SO(7,23)$ scalars into a $[SO(8) \times SO(24)]\backslash SO(8,24,\mathbb{R})$ symmetric manifold, acted upon by the U-duality group $SO(8,24,\mathbb{Z})$ [21]. To identify this subgroup of $SO(8,24)$ which remains as a quantum symmetry, it is useful to translate the action of the Weyl generator $g_{4H} \to 1/g_{4H}, l_H \to g_{4H}^2 l_H$ of the $Sl(2,\mathbb{Z})$ S-duality in $D = 4$ in terms of the three-dimensional variables using $1/g_{3H}^2 = V_7/(g_H^2 l_H^7)$. The result is an exchange of $1/g_{3H}^2$ and R_7/l_H, so that using the Weyl group of $Sl(7) \subset SO(7,23)$, we can transform $1/g_{3H}^2$ into R_i/l_H for any radius of T^7. This implies that we can think of the $SO(8,24)$ scalars as the moduli of an heterotic compactification on $T^7 \times S^1$, where the radius of S_1 is given by

$$R_8/l_H = 1/g_{3H}^2 \ . \tag{2.4}$$

This compact circle therefore appears as a dynamically generated dimension decompactifying at weak coupling. Note that this is *not* the usual M-theory direction, whose radius is instead $R_{11} = g_H l_H$. The dilatonic moduli parameterizing the Abelian part of $SO(8,24)$ are therefore

$$x_{i=1\ldots8} = (R_1/l_H, \ldots, R_7/l_H, 1/g_{3H}^2) \ . \tag{2.5}$$

We now turn to the dual compactifications, for which we only present the salient features, referring to [20] for a detailed treatment of the moduli spaces and their mapping to the heterotic theory. We also constrain ourselves to the case $d \geq 4$, in which case the heterotic theory on T^d is dual to type IIA on $K_3 \times T^{d-4}$, and further restrict to the $T^4/\mathbb{Z}_2$ orbifold point of K_3. Starting with $d = 4$, the duality map is given by [6]

$$l_H = g_{6IIA} l_{II} \ , \quad g_{6IIA} = \frac{1}{g_{6H}} \ , \quad \left(\frac{R_1}{l_H}\right)^2 = \frac{V_{K_3}}{l_{II}^4} \ , \tag{2.6}$$

which can be obtained by identifying the IIA NS5-brane on K_3 with the fundamental heterotic string, and the type IIA D0-brane with a heterotic Kaluza-Klein state along the circle of radius R_1 in T^4.

The relation between the heterotic and type II moduli can be obtained by studying the BPS spectrum. On the heterotic side, the BPS states are Kaluza-Klein and winding states

transforming as a *vector* of $SO(4, 4, \mathbb{Z})$, and possibly charged under the 16 $U(1)$ gauge fields. On the type IIA side, a set of BPS states is certainly given by the D0-, D2- and D4-branes wrapped on the even cycles of T^4, which are invariant under the $\mathbb{Z}_2$ involution. These states transform as a *conjugate spinor* of the T-duality group $SO(4, 4, \mathbb{Z})$, as D-branes should [4]. We thus find that the heterotic $g/l_{\mathrm{H}}^2, b$ and type IIA $G/l_{\mathrm{II}}^2, B$ moduli should be related by $SO(4, 4)$ *triality* [25, 20], which exchanges the vector and conjugate spinor representations. In particular, at the $T^4/\mathbb{Z}_2$ orbifold point with a square T^4, this reduces to

$$R_1|_{\mathrm{H}} = \sqrt{R_1 R_2 R_3 R_4}_{|\mathrm{IIA}} \ , \qquad R_i|_{\mathrm{H}} = \sqrt{\frac{R_1 R_i}{R_j R_k}}_{|\mathrm{IIA}} \ , \qquad i, j, k = 2, 3, 4 \ , \qquad (2.7)$$

where the radii are measured in the respective string length units. In logarithmic units, we recognize the standard triality matrix acting in the $SO(8)$ weight space.

There are also D2-brane states wrapped on the collapsed spheres at the sixteen orbifold singularities [26], and charged under the corresponding $U(1)$ fields. These are to be identified with the charged BPS states on the heterotic side, and their masses are matched by choosing the Wilson lines as [27]

$$y = \frac{1}{2} \begin{pmatrix} 0101 & 0101 & 0101 & 0101 \\ 0000 & 0000 & 1111 & 1111 \\ 0000 & 1111 & 0000 & 1111 \\ 0011 & 0011 & 0011 & 0011 \end{pmatrix} \ . \qquad (2.8)$$

This can also be derived by realizing that the Wilson lines along the first circle in T^4 map to the B-field fluxes on the collapsed two-spheres, which have been shown to be half a unit in order for the conformal field theory to be non-singular [28]. If we instead put this Wilson line to zero, we recover a gauge symmetry $SO(4)^8 = SU(2)^{16}$, as appropriate for the 16 A_1 singularities of $T^4/\mathbb{Z}_2$. This choice is relevant for M-theory compactified on K_3 at the $\mathbb{Z}_2$ orbifold point. If we further omit the Wilson lines on the 2nd (resp 2nd and 3rd) circles, the gauge symmetry is enlarged to $SO(8)^4$ (resp. $SO(16)^2$), which are relevant for F-theory on K_3 and type I' respectively. These relations explain why our results can easily been applied to these higher dimensional cases as well [20].

For the cases $d > 4$, we note that the T^{d-4} torus in the compactification of type IIA on $K_3 \times T^{d-4}$ is inert under heterotic-type II duality so that we can use the 6D duality map (2.6) to obtain the relation between this part of the compactification manifold on the two dual sides. For $d = 5$, this gives that the extra circles of compactification on the two dual sides are related by

$$R_{\mathrm{H}} = R_{\mathrm{A}}/g_{6\mathrm{IIA}} \ . \qquad (2.9)$$

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When $d = 6$, we obtain the mapping

$$S_\mathrm{H} = T_\mathrm{IIA} \ , \quad T_\mathrm{H} = S_\mathrm{IIA} \ , \quad U_\mathrm{H} = U_\mathrm{IIA} \tag{2.10}$$

between the four-dimensional couplings $S = a + i/g_4^2$, Kähler class T and complex structure U of the extra[‡1] T^2. Finally, for the $d = 7$ case, i.e. type IIA on $K_3 \times T^3$, the set of dilatonic moduli (2.5) of the $SO(8, 24)$ coset representative is mapped to

$$x_{i=1\ldots8} = \left(\sqrt{R_1 R_2 R_3 R_4}, \sqrt{\frac{R_1 R_2}{R_3 R_4}}, \sqrt{\frac{R_1 R_3}{R_2 R_4}}, \sqrt{\frac{R_1 R_4}{R_2 R_3}}, \frac{R_5}{g_\mathrm{6IIA}}, \frac{R_6}{g_\mathrm{6IIA}}, \frac{R_7}{g_\mathrm{6IIA}}, \frac{R_5 R_6 R_7}{g_\mathrm{6IIA}} \right) \ . \tag{2.11}$$

These moduli are not the standard $SO(4, 4) \times SO(3, 3)$ that arise from the reduction of the six-dimensional type II theory on $T^4/\mathbb{Z}_2 \times T^3$. The latter can however be reached by performing, in analogy with (2.7), an $SO(4, 4)$ triality acting on both $T^4/\mathbb{Z}_2$ moduli, and another $SO(4, 4)$ triality on the non-perturbative T^4 torus [22]. This gives the more familiar parameterization

$$y_{i=1\ldots8} = \left(R_1, \ldots, R_7, 1/g_\mathrm{3IIA}^2 \right) \ , \tag{2.12}$$

so that type IIA theory compactified on $K_3 \times T^3$ also appears to have a dynamically generated dimension, of size $\tilde{R}_8 = l_\mathrm{II}/g_\mathrm{3IIA}^2$.

3 Heterotic one-loop F^4 coupling and tree level IIA

Our first check of duality will be the matching of the F^4 couplings in $D = 6$ on the heterotic and type II side. On both sides there are 24 gauge fields. On the heterotic side, there are (0,16) vectors originating from the Cartan torus of the ten-dimensional gauge group and (4,0)+(0,4) from the reduction of the metric and antisymmetric tensor on T^4, corresponding to 4 vectors and 4 graviphotons. On the type IIA side, these gauge fields originate from the Ramond-Ramond sector. The 4 graviphotons can be understood as the reduction of the 4-form field strength in $D = 10$ on the 3 self-dual cycles of K_3 together with the ten-dimensional 2-form field-strength, whereas the 20 vector multiplets come from the 4-form in $D = 10$ on the 19 anti-self-dual cycles together with the 6-form field strength K_3 itself. At the $T^4/\mathbb{Z}_2$ orbifold point, the gauge fields split into (4,4) from the untwisted sectors and (0,16) (at each of the fixed points) from the twisted sectors.

[‡1]By further T-duality in one direction one may go to type IIB. For $d = 5$ this gives the SO(5,21) moduli of IIB/K_3 by decompactifying, while for $d = 6$ this induces $U \leftrightarrow T$ interchange in (2.10).

For simplicity we will focus here on the four-point amplitudes of the (0,16) vectors, referring to [20] for details on the other gauge fields. There are several arguments that support the fact that these F^4 couplings are non-renormalized [29, 30, 31, 15] in the heterotic string beyond one-loop for $D \geq 4$. Firstly, in ten dimensions these terms are related by supersymmetry to CP-odd couplings such as $B \wedge \text{Tr} F^4$, which should receive no corrections beyond one-loop for anomaly cancellation. A more explicit proof can be given at the level of string amplitudes [29]. Secondly, the only heterotic half-BPS instanton is the heterotic 5-brane, which needs a six-cycle to wrap in order to give a finite action instanton effect. So for $d < 6$ there can be no non-perturbative contributions beyond the one-loop result. Third, it is consistent with the factorization of the moduli space (2.1) and the T-duality symmetry $O(d, d+16, \mathbb{Z})$ to assume that $t_8 \text{Tr} F^4$ couplings are given at one-loop only and hence independent of the $\mathbb{R}^+$ factor. In $d = 6$ it is plausible that supersymmetry prevents the mixing of the $Sl(2, \mathbb{R})$ dilaton factor with the Narain moduli in F^4 couplings, in the same way as neutral hypermultiplets decouple from vector multiplets in $N = 2$ supergravity, and prevents corrections from NS5-brane instantons [32]. For $d = 7$, U-duality mixes the dilaton with the Narain moduli, so that a similar statement cannot hold. Gauge fields being Poincaré dual to scalars in 3 dimensions, the F^4 couplings translate into four-derivative scalar couplings, and should receive non-perturbative corrections, as we will see explicitly in Section 5.

Since the F^4 amplitude in toroidally compactified heterotic string theory is half-BPS saturated, the left-moving part of the four-gauge-boson (or graviton) amplitude merely provides the kinematic structure, whereas the right-moving currents reduce to their zero-mode part. The scattering amplitude between four vector multiplets at the one-loop level is given by the "elliptic genus" [33]

$$\Delta_{F^4}^{\text{Het,1-loop}} = l_{\text{H}}^{d-2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{p_R^4 \cdot Z_{d,d+16}(g/l_{\text{H}}^2, b, y)}{\bar{\eta}^{24}} \, , \qquad (3.1)$$

where we have stated the result for general d. Here l_{H} is the heterotic string length, and is reinstated on dimensional grounds. $Z_{d,d+16}$ denotes the partition function of the heterotic even self-dual lattice of signature $(d, d+16)$, parameterized by the metric g, Kalb-Ramond field b and Wilson lines y. p_R has modular weight $(0,1)$ and inserts the right-moving momentum corresponding to the choice of gauge boson F, and $1/\eta^{24} = 1/q + 24 + \ldots$ is the contribution of the 24 right-moving oscillators that generate the Hagedorn density of half-BPS states in the perturbative spectrum of the heterotic string. We remark here that amplitudes involving gravitons are similar with an insertion of an (almost) holomorphic form of weight (4,0), $\alpha E_4 + \beta \hat{E}_2^2$ instead of p_R^4. Amplitudes involving graviphotons have more powers of lattice momenta.

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Considering now the case $d = 4$, we find using the duality relations (2.6) with type IIA on K_3 and taking into account the particular normalization of the type II Ramond fields, that (3.1) translates into a tree-level type IIA result. On the other hand, it is still given by a modular integral on the fundamental domain of the upper-half plane, which is usually characteristic of one-loop amplitudes. The resolution of this paradox is that on the type IIA side, the gauge fields dual to the $(0, 16)$ heterotic ones originate from the twisted sectors of the orbifold. The correlator of four $\mathbb{Z}_2$ twist fields on the sphere can be re-expressed as the correlator of single-valued fields on the double cover of the sphere, which is a torus [34]; its modulus depends on the relative position of the four vertices, and hence should be integrated over, in qualitative agreement with the heterotic side. A careful computation yields the tree-level type IIA result

$$\Delta_{F^4}^{\text{IIA,tree}} = \frac{1}{g_{\text{II}}^2} g_{\text{II}}^4 \frac{l_{\text{II}}^6}{V_{K_3}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4}(G/l_{\text{II}}^2, B) \ , \tag{3.2}$$

where the factors of g_{II} correspond to the tree-level weight and the normalization of the Ramond fields respectively and G and B are the moduli of the T^4 on the IIA side. Here we have focused for simplicity on a particular choice of $(0, 16)$ fields: in general, (3.2) involves a shifted lattice sum integrated on a six-fold cover $\mathcal{F}_2$ of the fundamental domain $\mathcal{F}$. Note in particular, that the oscillators have dropped, in agreement with the fact that this amplitude should be half-BPS saturated. The normalization factor l_{II}^6/V_{K_3} has been chosen so as to agree with the heterotic result.

Still the type IIA result (3.2) is not quite of the same form as (3.1). For one thing, the type IIA result, being a half-BPS saturated coupling, does not involve any oscillators, in contrast to the heterotic side. For another, the $[SO(4) \times SO(4)] \backslash SO(4, 4, \mathbb{R})$ moduli $G/l_{\text{II}}^2, B$ are not the same as the heterotic $g/l_{\text{H}}^2, b$. In order to reconcile the two, we need two further ingredients.

Hecke identities. Since we are interested in comparing the dual theories at the orbifold point of K_3, we need to implement the choice of Wilson lines (2.8) on the heterotic side. These are best described [35] in terms of a $(\mathbb{Z}_2)^4$ freely acting orbifold, so that

$$Z_{4,20} = \frac{1}{2^4} \sum_{h,g} Z_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \bar{\Theta}_{16} \begin{bmatrix} h \\ g \end{bmatrix} \ , \tag{3.3}$$

where g and h run from 0 to 15 and are best seen as four-digit binary numbers; h labels the twisted sector while the summation over g implements the orbifold projection in that sector. The blocks $Z_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}$ are partition functions of (4,4) lattices with half-integer shifts, and $\bar{\Theta}_{16} \begin{bmatrix} h \\ g \end{bmatrix}$ are antiholomorphic conformal characters. The operator $\bar{Q}^4 = p_R^4$ only acts on the latter. Using techniques first developed in [36], it can be shown [18, 20] that the

conformal blocks $\Phi \begin{bmatrix} h \\ g \end{bmatrix} = Q^4 \Theta_{16} \begin{bmatrix} h \\ g \end{bmatrix} / \eta^{24}$ occurring in the modular integral can be replaced by two-thirds their image λ under the Hecke operator

$$H_{\Gamma_2^-} . \Phi(\tau) = \frac{1}{2} \left(\Phi\left(-\frac{1}{2\tau}\right) + \Phi\left(\frac{\tau}{2}\right) + \Phi\left(\frac{\tau+1}{2}\right) \right) \tag{3.4}$$

provided this image is a constant real number:

$$H_{\Gamma_2^-} \cdot \left[\frac{Q^4 \Theta_{16} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\eta^{24}} \right] = \lambda \; . \tag{3.5}$$

The relation (3.5) indeed holds for all the conformal blocks of interest in this construction. The modular integral thus reduces to

$$A_{F^4}^{\text{Het}} = l_H^2 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4}(g/l_H^2, b) \tag{3.6}$$

and the Hagedorn density of half-BPS states in (3.1) has thus cancelled.

Triality. The last step needed to identify the type IIA and heterotic result is to understand how the triality transformation (2.7) relating the moduli on the dual sides, equates the integrals of the partition function $Z_{4,4}(g/l_H^2, b)$ and $Z_{4,4}(G/l_{II}^2, B)$ on the fundamental domain of the upper-half plane. It is easy to convince oneself that such an equality cannot hold at the level of integrands, by looking at some decompactification limits. However, it has been shown that such modular integrals could be represented either as Eisenstein series for the T-duality group $SO(4, 4, \mathbb{Z})$, in the *vector* or (conjugate) *spinor* representations, thanks to the identity [12]:

$$\pi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4} = \mathcal{E}_{\mathbf{V};s=1}^{SO(4,4,\mathbb{Z})} = \mathcal{E}_{\mathbf{S};s=1}^{SO(4,4,\mathbb{Z})} = \mathcal{E}_{\mathbf{C};s=1}^{SO(4,4,\mathbb{Z})} \; . \tag{3.7}$$

This implies the invariance of the modular integral of $Z_{4,4}(g/l_s^2, b)$ under triality transformation of the moduli, which completes the argument [20] that in six dimensions the one-loop F^4 heterotic coupling equals the tree-level F^4 coupling in IIA.

In the above, we have focused mainly on the case of four identical gauge fields in (0,16). Similarly, one can consider two groups of two identical gauge fields or four distinct ones. Also in these cases can one make a successful comparison between the heterotic and type IIA side, though a precise match requires the exact identification of the gauge fields on either side, which is still an open problem. It would be also interesting to understand how the duality holds at other orbifold points of K_3, since naively the correlator of $\mathbb{Z}_n$ twist fields on the sphere involves higher genus Riemann surfaces, albeit of a very symmetric type.

4 D-instantons and NS5-brane instantons in IIA

Having reproduced the type IIA tree-level F^4 coupling in 6 dimensions from the heterotic one-loop amplitude, we now would like to use the duality map in lower dimensions to show some non-trivial consequences of string-string duality. The absence of non-perturbative corrections to F^4 on the type IIA side in $D = 6$ is easily understood as follows: Such instanton effects should arise from half-BPS solitonic solutions, with their entire (Euclidean) world-volume wrapped on supersymmetric cycles of the same dimensionality. Since, type IIA has only even D-branes, these should wrap odd-dimensional supersymmetric cycles, which are not present in K_3, which has only 0,2 and 4-cycles. Moreover, NS5-brane (KK5-brane) instantons would require six- (seven-) cycles, and hence cannot occur until $D = 4$ ($D = 3$). On the other hand, this implies that for compactification on $K_3 \times T^{d-4}$, with $d > 4$, we would expect instanton contributions on these general grounds. In particular, we expect for increasing d the following instantons to appear:

- $d = 5$: D0,D2,D4-brane, wrapping 0,2,4-cycle in K_3 and S^1
- $d = 6$: NS5-brane, wrapping 4-cycle in K_3 and T^2
- $d = 7$: D6,KK5-brane, wrapping 4-cycle in K_3 and T^3

In this section, we explicitly derive these effects from the one-loop heterotic F^4 coupling for the first two cases[‡2] above, leaving the last case to Section 5

The general approach to quantitatively understand these instanton configurations from the 1-loop heterotic result at the orbifold point

$$\Delta_{F^4}^{\mathrm{Het}} = l_{\mathrm{H}}^{d-2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{d,d}(g/l_{\mathrm{H}}^2, b) , \tag{4.1}$$

is as follows. First, using the duality map we find that weak coupling expansion on the IIA side corresponds to a large volume expansion of T^{d-4} on the heterotic side. At the level of the toroidal lattice sum $Z_{d,d}$, this large volume expansion is best exhibited by adopting a Hamiltonian representation for the $Z_{4,4}$ part and a Lagrangean representation for the remaining $Z_{d-4,d-4}$ part. Assuming for simplicity zero Wilson lines of the 6D-gauge fields around the T^{d-4}, we thus decompose

$$Z_{d,d} = Z_{4,4} Z_{d-4,d-4} , \qquad Z_{4,4} = \sum_{m_i, n^i} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} , \qquad q = e^{2\pi i \tau} , \tag{4.2}$$

$$Z_{d-4,d-4} = V_{d-4} \sum_{p^\alpha, q^\alpha} e^{-\frac{\pi}{\tau_2}(p^\alpha + \tau q^\alpha)(g_{\alpha\beta} + b_{\alpha\beta})(p^\beta + \bar{\tau} q^\beta)} . \tag{4.3}$$

[‡2]We note here that, by the same reasoning, in the type IIB picture instanton configurations already exist in $D = 6$, since D-1,D1 and D3 branes can wrap the 0,2 and 4-cycles, as first computed in [17]. See also [20] for the $SO(5, 21, \mathbb{Z})$ U-duality invariant $t_{12}H^4$ coupling in that case.

Here, m_i, n^i denote the momenta and windings on T^4 and $2p^2_{L,R} = m^t(M_{4,4} \pm \eta)m$ with $m = (m_i, n^i)$, and $M_{4,4}$ the $SO(4,4)$ moduli matrix. Then we use the method of orbits on the last factor to evaluate the modular integral as a sum over orbits, where each term in the orbit decomposition will be a distinct term in the large T^{d-4} volume expansion. In particular, this shows a (finite) power series in the volume, corresponding to perturbative contributions on the dual side. The interesting part consists of the terms that are exponentially suppressed and are due to world-sheet instantons on the heterotic side, which map to space-time instantons on the type IIA side.

Let us see first how this applies to $D = 5$, in which case it follows from the duality relation (2.9) that the weak coupling regime on the type IIA side corresponds to the large $R_{\rm H}$ expansion on the heterotic side. According to (4.2), (4.3) the 5D-coupling can thus be written as

$$\Delta_{5D} = l_{\rm H}^2 R_{\rm H} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{p,q} \sum_{m_i,n^i} \exp\left(-\pi \frac{R_{\rm H}^2 |p - \tau q|^2}{l_{\rm H}^2 \tau_2}\right) \tau_2^2 q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} . \tag{4.4}$$

We now apply the standard orbit decomposition method on the integers (p,q), trading the sum over $Sl(2,\mathbb{Z})$ images of (p,q) for a sum over images of the fundamental domain $\mathcal{F}$ [37] (see [16, 12] for relevant formulae). The zero orbit gives back the six-dimensional result (3.2) up to a volume factor, and reproduces the tree-level type II contribution in 5 dimensions:

$$\Delta_{5D}^{\rm zero} = l_{\rm H}^2 R_{\rm H} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4}(g/l_{\rm H}^2, b) = R_{\rm A} \Delta_{\rm IIA,4D}^{\rm tree} . \tag{4.5}$$

The degenerate orbit on the other hand, with representatives $(p,0)$, can be unfolded onto the strip $|\tau_1| < 1/2$. The τ_1 integral then imposes the level matching condition $p_L^2 - p_R^2 = 2m_i n^i = 0$, and the τ_2 integral can be carried out in terms of Bessel functions. In type IIA variables the result is

$$\Delta_{5D}^{\rm deg} = 2 g_{\rm 6IIA} l_{\rm II} R_{\rm A}^2 \sum_{p \neq 0} \sum_{(m_i, n^i) \neq 0} \delta(m_i n^i) \cdot \frac{|p|}{\sqrt{m^t M_{4,4} m}} K_1\left(2\pi \frac{R_{\rm A}}{g_{\rm 6IIA} l_{\rm II}} |p| \sqrt{m^t M_{4,4} m}\right) , \tag{4.6}$$

where $M_{4,4}$ is now the mass matrix of D-brane states wrapped on the untwisted cycles of $T^4/\mathbb{Z}_2$. Given the asymptotic behaviour $K_1(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$, we see that this is a sum of order e^{-1/g_s} non-perturbative effects corresponding to $N = pr$ Euclidean (anti) D-branes wrapped on S_1 times a cycle of homology charges $(m_i, n^i)/r$ on T^4, where r is the greatest common divisor of (m_i, n^i)[‡3].

[‡3]For general choices of twisted gauge fields, the actual answers involve a torus partition function $Z_{4,4}\begin{bmatrix} h \\ g \end{bmatrix}$ with half integer shifts. This corresponds to open Euclidean D-branes with half integer wrappings number, as required in order to interpolate between different fixed points.

It is worth pointing out a number of peculiarities of the result (4.6). First, due to the absence of a holomorphic insertion in (4.1), all instanton effects are due to *untwisted* D-branes wrapped along even cycles of K_3, even though we are discussing F^4 couplings between fields located on the fixed points of the orbifold. This is in contrast to the result in four-derivative scalar couplings [17], where a contribution from the whole Hagedorn density of BPS states was found. This is an important simplification due to our choice of the orbifold point in the K_3 moduli space. Second, the integration measure corresponding to a given number of D-branes N is easily seen to be $\sum_{r|N}(1/r^2)$, where r runs over the divisors of N, just as in the case of D-instanton effects in theories with 32 supersymmetries [7, 38]. This is a somewhat unexpected result, since the bulk contribution to the index for the quantum mechanics with 8 unbroken symmetries is $1/N^2$ instead [39], which did arise in four-derivative scalar couplings at the enhanced symmetry point [17, 40].

Turning to the next case, $D = 4$, we see from the duality map (2.10) that weak coupling in type IIA corresponds to the limit where the heterotic T^2 decompactifies. The study of this decompactification limit proceeds as outlined in the general strategy above. Performing an orbit decomposition on the integers running in the Lagrangean representation of the T^2 lattice, the zero orbit and degenerate orbit reproduce the tree-level and D-instanton contributions on the type II side. The novelty in this case is that there is a third orbit, namely the non-degenerate orbit, which contributes as well. The integral on τ_1 is Gaussian, and the subsequent integral along τ_2 is again given by a Bessel function. Before carrying out this integration exactly, it is more enlightening to determine the saddle point, which controls the instanton effects at leading order. Using (4.2), (4.3) with $d = 6$, the saddle point equations are easily found to be

$$q^\alpha g_{\alpha\beta}(p^\beta - \tau_1 q^\beta) + i\tau_2 m_i n^i = 0 \ , \tag{4.7}$$

$$-(p^\alpha - \tau_1 q^\alpha)g_{\alpha\beta}(p^\beta - \tau_1 q^\beta) + \tau_2^2(q^\alpha g_{\alpha\beta}q^\beta + m^t M_{4,4}m) = 0 \ , \tag{4.8}$$

where p^α and q^α are the integers running in the T^2 lattice partition function, and should be summed over $Sl(2,\mathbb{Z})$ orbits such that $p^1 q^2 - p^2 q^1 \neq 0$ only. $g_{\alpha\beta}$ is the metric on T^2 in heterotic units. These equations are easily solved and after translation to type IIA variables correspond to a classical action

$$S_{\mathrm{cl}} = 2\pi\sqrt{\frac{p^2 q^2 - (pq)^2}{(q^2)^2}\left(\frac{(q^2)^2}{g_{6\mathrm{IIA}}^4 l_{\mathrm{II}}^4} + \frac{q^2 m^t M_{4,4}m}{g_{6\mathrm{IIA}}^2 l_{\mathrm{II}}^2} + (m_i n^i)^2\right)} + 2\pi i\left(\frac{(pq)(m_i n^i)}{q^2} + pBq\right) \ . \tag{4.9}$$

In particular, the real part of the classical action scales as $1/g_{6\mathrm{IIA}}^2$. The corresponding non-perturbative effects should therefore be interpreted as coming from $N = |p^1 q^2 - p^2 q^1|$

NS5-branes wrapped on $K_3 \times T^2$, and bound to D-brane states wrapped on an even cycle of K_3 times a circle on T^2 determined by the integers q^1, q^2.

The explicit result of the τ integration gives

$$\Delta_{4D}^{\text{n.d.}} = 4l_{\text{H}}^4 \sum_{p^\alpha, q^\alpha} \sum_{m_i, n^i} \left(\frac{(q^2)^2 + q^2 m^t M_{4,4} m + (m_i n^i)^2}{p^2 q^2 - (pq)^2} \right)^{3/4} K_{3/2} \left(\Re S_{\text{cl}} \right) e^{i \Im S_{\text{cl}}} . \qquad (4.10)$$

In particular, we may look at the contribution of pure NS5-brane instantons, corresponding to $m_i = n^i = 0$. Choosing an orbit representatives and using the exact expression for the Bessel function

$$K_{3/2}(x) = \sqrt{\pi/2x} \, (1 + 1/x) \, e^{-x} \qquad (4.11)$$

we obtain

$$\Delta_{4D}^{\text{NS5}} = 2(g_{6\text{IIA}} l_{\text{II}})^4 U_2 \sum_N \mu(N) \left(N + \frac{1}{2\pi S_2} \right) e^{-2\pi N S_2} \left(e^{2\pi i N S_1} + e^{-2\pi i N S_1} \right) , \qquad (4.12)$$

where we used the type II variable $S = a + iV_{K_3} V_{T^2}/(g_{\text{II}}^2 l_{\text{II}}^6)$ and extracted the instanton measure [20]

$$\mu(N) = \sum_{r|N} \frac{1}{r^3} , \qquad (\text{NS5-brane on } K_3 \times T^2) . \qquad (4.13)$$

This result gives a prediction for the index (or rather the bulk contribution thereto) of the world-volume theory of the type II NS5-brane wrapped on $K_3 \times T^2$. It is a challenging problem to try and derive this result from first principles. It is also remarkable that, in virtue of (4.11) and in contrast to D-instantons, the NS5-instantons contributions do not seem to receive any perturbative subcorrections beyond one-loop.

It is interesting to compare this result to the corresponding index of the heterotic 5-brane wrapped on T^6, which can be extracted from the non-perturbative R^2 couplings in the heterotic string compactified on T^6 [41, 32]. Those can be computed by duality from the one-loop exact R^2 couplings in type II on $K_3 \times T^2$ [13], and read

$$\Delta_{R^2} = \hat{\mathcal{E}}_{\mathbf{2};s=1}^{Sl(2,\mathbb{Z})} = -\pi \log(S_2 |\eta(S)|^4) = \frac{\pi^2}{3} S_2 + 2\pi \sqrt{S_2} \sum_N \mu(N) e^{-2\pi N S_2} \left(e^{2\pi i N S_1} + e^{-2\pi i N S_1} \right) . \qquad (4.14)$$

The summation measure turns out to be different from (4.13) and given instead by

$$\mu(N) = \sum_{r|N} \frac{1}{r} , \qquad (\text{Het 5-brane on } T^6) . \qquad (4.15)$$

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It is also worthwhile to notice that there are no subleading corrections around the instanton in the heterotic 5-brane case. This is in contrast to D-instantons, for which the saddle point approximation to the Bessel function K_1 is not exact. It would be interesting to have a deeper understanding of these non-renormalization properties, possibly using the CFT description of the 5-brane [42].

5 Heterotic/type II duality in $D = 3$

We finally present the three-dimensional case, that is the duality between heterotic on T^7 and type IIA on $K_3 \times T^3$. As shown in Section 2, this case has the interesting feature that there is an $SO(8, 24, \mathbb{Z})$ U-duality symmetry [21] on each dual side, which can be used to conjecture a non-perturbative F^4 amplitude [22] on both the heterotic as well as the type IIA side, while in $D > 3$ this amplitude was one-loop exact. The proposed U-duality invariant F^4 amplitude reproduces the known one-loop and tree-level answers on the heterotic and type II sides respectively. Moreover, this amplitude will contain instanton corrections along with the perturbative contributions, and hence give us information in particular about the elusive heterotic instantons. Another property is that string theories in three dimensions possess new instanton configurations that were not present in $D > 3$, namely the Kaluza-Klein 5-brane instantons. These instantons are obtained by tensoring a Taub-NUT gravitational instanton asymptotic to $\mathbb{R}^3 \times S^1$ with a flat T^6, where $T^7 = S^1 \times T^6$. They are the ten-dimensional Euclidean version [2] of the Kaluza-Klein monopoles introduced in [43]. Moreover, $D = 3$ is just one step away from $D = 2$, where the U-duality group becomes infinite-dimensional, and predicts an infinite set of particles with exotic dependence $1/g_s^{n+3}$ on the coupling [44, 4]. Similar states already appear as particles in $D = 3$ with mass $1/g_s^3$. Instantons in $D = 3$ are however free of these infrared problems, and the study of exact amplitudes in $D = 3$ may shed light on the non-perturbative spectrum.

Our starting point is the heterotic one-loop F^4 amplitude in three dimensions, given by (3.1) with $d = 7$, which already incorporates most of the symmetries, namely the T-duality $SO(7, 23, \mathbb{Z})$. First, we dualize the vectors into scalars, which is achieved by adding a Lagrange multiplier term $\phi_a dF^a$ to the gauge kinetic term $(l_H/g_{3H}^2)F^a(M_{7,23})_{ab}^{-1}F^b$ and integrating out the field strength F^a, yielding $F^a = \frac{g_{3H}^2}{l_H}M_{7,23}^{ab}d\phi_b$. The resulting one-loop result is then

$$\Delta^{\text{Het},1-\text{loop}}_{(\partial\phi)^4} = l_H g_{3H}^8 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{p_R^4}{\bar{\eta}^{24}} Z_{7,23}\, t_4(M_{7,23}\partial\phi)^4 \,, \tag{5.1}$$

where t_4 is the tensor t_8 with pairs of indices raised with the ϵ_3 antisymmetric tensor. This result can now be covariantized under U-duality, leading to the proposal that the exact

$(\partial\phi)^4$ threshold in heterotic string on T^7, or any of its dual formulations, is given by

$$I_{(\partial\phi)^4} = l_{\mathrm{P}} \int d^3x \sqrt{g} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{Z_{8,24}(g/l_{\mathrm{H}}^2, b, \phi, g_{3\mathrm{H}}^2)}{\bar{\eta}^{24}} \ t_8[(e_{8,24}^{-1}\partial_\mu e_{8,24})_{ia} \ p_R^a]^4 \ , \qquad (5.2)$$

where $l_{\mathrm{P}} = g_{3\mathrm{H}}^2 l_{\mathrm{H}}$ is the three-dimensional (U-duality invariant) Planck length. In this expression, $Z_{8,24}$ is the Theta function of the non-perturbative $\Gamma_{8,24}$ lattice, which is invariant under both U-duality $SO(8, 24, \mathbb{Z})$ and $Sl(2, \mathbb{Z})$ modular transformations of τ. $e_{8,24}^{-1}\partial_\mu e_{8,24}$ is the left-invariant one-form on the coset $[SO(8) \times SO(24)]\backslash SO(8, 24)$, pulled-back to space-time (see [22] for further details). This structure is also the one that arises in a one-loop four-scalar amplitude as shown in [17]. The conjecture (5.2) satisfies the following criteria:

• It is $SO(8, 24, \mathbb{Z})$ invariant by construction;

• It correctly reproduces the heterotic 1-loop coupling;

• The non-perturbative contributions come from heterotic 5-branes and KK5-branes, which are the expected ones in $D = 3$;

• The result decompactifies to the known purely perturbative result in $D \geq 4$;

• Via heterotic/type II and heterotic/type I duality, the corresponding amplitude in type II and I shows the correct perturbative terms and the expected instanton corrections.

To prove these claims, we will restrict ourselves as in Section 4 to the particular subspace of moduli space, corresponding to the $T^4/\mathbb{Z}_2 \times T^3$ orbifold point on the type II side. As in Section 3, for that choice of Borel moduli, the Dedekind function $\bar{\eta}^{24}$ in (5.2) cancels against the action of p_R^4 on the D_{16} part of the lattice, and we are left with the simpler expression

$$\Delta_{(\partial\phi)^4} = l_{\mathrm{P}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{8,8}(g/l_{\mathrm{H}}^2, b, g_{3\mathrm{H}}^2) \ , \qquad (5.3)$$

where the partition function $Z_{8,8}$ now runs over the lattice $\Gamma_{1,1}^8$ only. In proving the assertions above, we will again make extensive use of the weak-coupling expansion method outlined at the beginning of Section 4.

Heterotic instanton expansion. We first notice that defining $1/g_{3\mathrm{H}}^2 = R_8/l_{\mathrm{H}}$ as in (2.4), the weak coupling expansion of the result (5.3) becomes a large R_8 expansion. Consequently, we adopt a Lagrangean representation for the S^1 part and a Hamiltonian representation for the remainder:

$$\Delta_{(\partial\phi)^4} = l_{\mathrm{P}} \frac{R_8}{l_{\mathrm{H}}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{p,q} \sum_v \exp\left(-\pi \frac{R_8^2 |p - \tau q|^2}{l_{\mathrm{H}}^2 \tau_2}\right) \tau_2^{7/2} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} \ , \qquad (5.4)$$

where $v = (m_i, n^i)$ now denotes the $7 + 7$ perturbative momenta and windings. Applying the same orbit decomposition as below Eq.(4.4), we find that the zero orbit gives back the

perturbative result (5.1)

$$\Delta^{\text{zero}}_{(\partial\phi)^4} = l_{\text{H}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{7,7} \ , \tag{5.5}$$

while the degenerate orbit gives

$$\Delta^{\text{deg}}_{(\partial\phi)^4} = 2l_{\text{H}} \sum_{p\neq0} \sum_{v\neq0} \delta(v^t\eta_{7,7}v) \left(\frac{p^2}{g_{3\text{H}}^4 v^t M_{7,7} v}\right)^{5/4} K_{5/2}\left(\frac{2\pi}{g_{3\text{H}}^2}|p|\sqrt{v^t M_{7,7} v}\right) \ . \tag{5.6}$$

From the expansion of the Bessel function

$$K_{5/2}(x) = \sqrt{\pi/2x}[1 + 3/x + 3/x^2]e^{-x} \tag{5.7}$$

we see that these are non-perturbative contributions with classical action

$$\Re(S_{\text{cl}}) = \frac{2\pi}{g_{3\text{H}}^2}|p|\sqrt{v^t M_{7,7} v} \ . \tag{5.8}$$

Choosing for v either "momentum" charges or "winding" charges, we find an action

$$\frac{1}{g_{3\text{H}}^2}\frac{l_{\text{H}}}{R_i} = \frac{V_6}{g_{\text{H}}^2 l_{\text{H}}^6} \ , \qquad \frac{1}{g_{3\text{H}}^2}\frac{R_i}{l_{\text{H}}} = \frac{V_6 R_i^2}{g_{\text{H}}^2 l_{\text{H}}^8} \ , \tag{5.9}$$

which identifies these instantons as heterotic 5 branes and KK5-branes respectively, wrapped on a T^6 inside T^7. The summation measure for these effects is easily extracted, and yields

$$\mu_{\text{Het}}(N) = \sum_{d|N} \frac{1}{d^5} \ , \tag{5.10}$$

where $N = gcd(p, m_i, n^i)$. We also note that the Bessel function $K_{5/2}$ in (5.7) exhibits only two subleading terms beyond the saddle point approximation, so that these instanton contributions do not receive any corrections beyond two loops.

Since NS5-brane instantons appear in the three-dimensional $(\partial\phi)^4$ result, one may wonder how they can not contribute in four-dimensions, where the F^4 threshold has been argued to be purely one-loop [32, 20]. To verify this, one decomposes $Z_{8,8} = Z_{2,2}Z_{6,6}$, where $Z_{2,2}$ stands for the lattice sum on the non-perturbative two torus in the (7,8) direction and $Z_{6,6}$ is the lattice sum for the perturbative states in $D = 4$. Decompactifying to four dimensions then corresponds to a large volume limit of the two-torus. The orbit decomposition then shows that only the trivial orbit contributes in the decompactification limit, corresponding to the $D = 4$ one-loop result, while both the degenerate and non-degenerate orbits contain exponentially suppressed terms that vanish.

Type II instanton expansion. We finally consider the type II interpretation of the conjecture (5.2). From the moduli identification (2.11), it is seen that the weak coupling expansion on the type II side corresponds to the large volume expansion of the non-perturbative T^4 in the 5–8 directions, with volume

$$v_4 = \frac{V_3^2}{g_{6\mathrm{IIA}}^4 l_{\mathrm{II}}^6} \, , \tag{5.11}$$

where $V_3 = R_5 R_6 R_7$. Using the by now familiar method, we thus decompose $Z_{8,8} = Z_{4,4}^{(1-4)} Z_{4,4}^{(5-8)}$ with Hamiltonian and Lagrangean representation for the two factors respectively. For use below, we define the rescaled metric G_4 on the non-perturbative four-torus

$$G_4 = g_{6\mathrm{IIA}}^2 g_4 \equiv e_4^t e_4 \, , \qquad e_4 = \mathrm{diag}(R_I/l_{\mathrm{H}}, V_3/l_{\mathrm{H}}^3) \, , \qquad I = 5, 6, 7 \, , \tag{5.12}$$

which depends on the geometric moduli only.

We can now perform again an orbit decomposition (see [10, 16] for a discussion of the orbit decomposition for $Z_{d,d}$, $d > 2$, which generalizes the $d = 2$ case [37].) Then, the trivial orbit gives

$$\Delta_{(\partial\phi)^4}^{\mathrm{zero}} = g_{3\mathrm{IIA}}^2 l_{\mathrm{II}} \frac{V_3^2}{g_{6\mathrm{IIA}}^4} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4} = \frac{V_3}{(g_{6\mathrm{IIA}} l_{\mathrm{II}})^4} \Delta_{F^4}^{\mathrm{tree}, 6D} \, , \tag{5.13}$$

which, using (2.6) shows the correct 3D tree-level result, directly induced from the 6D tree-level F^4 coupling (3.2). The degenerate orbit is evaluated as

$$\Delta_{(\partial\phi)^4}^{\mathrm{deg}} = 2 \frac{V_3}{g_{6\mathrm{IIA}}^2 l_{\mathrm{II}}^2} \sum_{p^\alpha \neq 0} \sum_{(m_i, n^i) \neq 0} \delta(m_i n^i) \frac{1}{g_{6\mathrm{IIA}}} \frac{\sqrt{|p^t G_4 p|}}{\sqrt{m^t M_{4,4} m}} K_1 \left(\frac{2\pi}{g_{6\mathrm{IIA}}} \sqrt{p^t G_4 p} \sqrt{m^t M_{4,4} m} \right) \, , \tag{5.14}$$

which generalizes (4.6). In particular, from the argument of the Bessel function K_1, we recognize the contributions of Euclidean D-branes wrapped on an even cycle of K_3, times a one-cycle of T^3 (for p in the 5, 6, 7 directions of the non-perturbative torus), or the whole T^3 for p in the 8th direction. The latter case corresponds to the contributions of Euclidean D6-branes, which start contributing in three dimensions.

For the non-degenerate orbit, the integral is dominated by a saddle point of the same form as in (4.7), (4.8), with the two torus metric g replaced by the non-perturbative four-torus metric. Then the final integral can be expressed again in terms of the $K_{3/2}(\Re S_{\mathrm{cl}})$ Bessel function as in (4.10), where $\Re S_{\mathrm{cl}}$ is exactly of the same form as in (4.9), with the distinction that all inner products of p, q vectors are taken with the metric G_4 defined in (5.12). In particular, setting $m_i = n^i = 0$ and switching on one charge at a time for simplicity, we identify these non-perturbative effects as coming from NS5-brane instantons

	IIA/$K_3 \times T^{d-4}$	Het/T^d	I/T^d
Dp	2 (F^4)		D1: 0 (F^4) D5: 1 (R^2), 5 (F^4)
NS5	3 (F^4)	1 (R^2), 5 (F^4)	
KK5	3 (F^4)	5 (F^4)	5 (F^4)

Table 6.1: Instanton contributions to F^4 and R^2 couplings in theories with 16 supersymmetries. The entry denotes the exponent r appearing in the summation measure $\sum_{d|N} d^{-r}$.

(4.12) as found already in $D = 4$. However, there are in addition genuine three-dimensional effects, corresponding to KK5-brane instantons, with action

$$\frac{V_3 R_K}{g_{6IIA}^2 l_{II}^4} = \frac{V_{K_3} R_I R_J R_K^2}{g_{II}^2 l_{II}^8} \ . \tag{5.15}$$

For general values of the charges p, q, m, we obtain contributions from boundstates of NS5 and KK5-branes with D-branes. It is a simple matter to derive the summation measure for N KK5-brane instantons in type IIA/$K_3 \times T^3$, which turns out to be identical to the one derived for NS5-brane instantons in IIA/$K_3 \times T^2$, given in (4.13).

6   Conclusion and Outlook

The F^4 couplings in theories with 16 supersymmetries provide a rich setting in which to perform non-trivial tests of string-string duality and compute new instanton effects in string theory. We have focused in these lectures on these amplitudes in toroidally compactified heterotic theory and its dual type IIA compactifications in $6 \leq D \leq 3$. In fact, in the lowest dimensional case $D = 3$, one finds half-BPS instanton effects arising from all Dp, NS5 and KK5-branes that are present in these theories. We emphasize that this case summarizes all other new results as the other cases can be reached by decompactification, but for pedagogical reasons the simpler cases were discussed first.

Many other interesting results follow by considering the higher dimensional cases [20]: In $D = 7$, the M-theory four-gluon amplitude for $SU(2)$ gauge bosons located at the A_1 singularities of K_3 was obtained. In $D = 8$, one recovers the F^4 amplitude for $SO(8)$ gauge bosons located at the orientifold planes of Sen's F-theory model [45], also considered in [18]. In $D = 9$, the F^4 couplings at the $SO(16) \times SO(16)$ point were computed, and shown not to involve the higher genus contact contributions found in [15, 16]. Although we have hardly mentioned it, the type I picture is very similar to the heterotic one: Using heterotic/type

I duality [5], the heterotic one-loop amplitude reproduces the familiar disk and cylinder amplitudes, together with D1-instanton effects. The heterotic 5-brane and KK5-brane contributions turn into type I D5-branes and KK5-branes, and their summation measure is unaffected by the duality. The table above summarizes the instanton summation measures for all cases known so far. It is a challenging problem to rederive the measures for the NS5-brane and KK5-instantons.

Among the further open problems we mention just a few: We have generally focused on instanton contributions coming from bound states of a singly type of brane, but from our results it should be possible to extract more detailed information pertaining complicated bound states of various distinct branes. Understanding BPS-amplitudes that break more than half of the supersymmetries is expected to provide further insights into instanton effects, as well as interesting generalizations of the automorphic forms that govern half-BPS amplitudes. Another important direction is half-BPS amplitudes in theories with yet lower supersymmetry, and computing instanton generated superpotentials in N=1 theories. Finally, an interesting challenge should be to consider further compactification to two-dimensional supergravity, in which case the U-duality symmetry is enhanced to an affine symmetry.

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